On the Vertex Operators of the Elliptic Quantum Algebra $U_{q,p}(\widehat{sl_2})_k$

Wen-Jing Chang [a,b] **and Xiang-Mao Ding** [a1]

[a] *Institute of Applied Mathematics,*
Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, P.O.Box 2734, Beijing, 100190, P.R. China.
[b] *Graduate School of Chinese Academy of Sciences, China.*

Abstract

A realization of the elliptic quantum algebra $U_{q,p}(\widehat{sl_2})$ for any given level k is constructed in terms of three free boson fields and their accompanying twisted partners. It can be viewed as the elliptic deformation of Wakimoto realization. Two screening currents are constructed; they commute or anti-commute with $U_{q,p}(\widehat{sl_2})$ modulo total q-differences. The free fields realization for two types vertex operators nominated as the type *I* and the type *II* vertex operators are presented. The twisted version of the two types vertex operators are also obtained. They all play crucial roles in calculating correlation functions.

1   Introduction

Infinite-dimensional symmetries, such as Virasoro algebra (*W*-algebra, for more general) and affine Lie algebra play a central role in the two-dimensional Conformal Field Theories (2D

CFT) [1]. While for the non-conformal (off-critical) integrable theory, their roles are taken over by the so called quantum algebras. From the algebraic point of view, there are three kinds of quantum algebras, according to their different exchange properties, which are nominated as rational, trigonometric and elliptic quantum algebras respectively. The quantum algebras of the former two kinds could be regarded as certain degenerate cases of the last one. For example, the quantum affine algebra (trigonometric), which is also known as the quantum group, and the Yangian double with central (rational) could be obtained as a certain limited case for the elliptic quantum algebras. Various versions of elliptic quantum algebras, also known as elliptic quantum groups [2, 3, 4] have been introduced, through an attempt to understand elliptic face models of statistical mechanics, and in its semiclassical limit, a CFT of Wess-Zumino-Witten (WZW) models on tori. Their roles are similar to the Kac-Moody algebras in WZW models. From Hopf algebra point of view, the elliptic quantum groups are nothing but quantum affine algebras equipped with a co-product different from the original one by a certain kind of twisting, so they can be viewed as quasi-Hopf algebras in the sense of Drinfeld [5]. There are two types of elliptic quantum groups which correspond to different types of integrable models: the vertex type $A_{q,p}(\widehat{sl_N})$ and the face type $B_{q,\lambda}(\mathcal{G})$, where $\mathcal{G}$ is a Kac-Moody algebra associated with a symmetrizable generalized Cartan matrix [6]. The former is closely related to vertex models, for example, the XYZ model, or equivalently, the eight vertex model in the principal regime [7]; while some face models, such as the Andrew-Baxter-Forrester (ABF) models [8] which are 'solid-on-solid' (SOS) face models, possess symmetries corresponding to the face type elliptic algebras.

From mathematical, it is natural to study these algebraic objects' structures and their representations. In physical applications, their representations are also required. The standard scheme to study integrable models in field theories or statistical mechanics is solving the following basic problems: to diagonalize the given Hamiltonian, and then to compute the correlation functions. Usually, it is quite difficult to solve such problems directly. While it has been indicated that the algebraic analysis method is an extremely powerful tool in studying solvable lattice models, especially in deriving the correlation functions. This method is based on the infinite dimensional quantum group symmetry possessed by a model and the representation theory of such symmetry. As a result, if one expect to perform algebraic analysis of the above both types of elliptic lattice models, he should firstly study the corresponding elliptic quantum groups and their representations.

In practice, the Wakimoto realization, or the so called free field method, which is an infinite dimensional extension of the Heisenberg algebra, is a quite effective and useful approach to study complicated algebraic structures and their representations. The well known example is the realization of the affine Lie algebras [9, 10, 11]. It is also a common method to obtain

representations of quantum affine algebras [12] and Yangian double [13]. In [14, 15], the XXZ model in the anti-ferromagnetic regime was solved by applying the level one representation theory of the quantum affine algebra $U_q(\widehat{sl_2})$. In studying higher spin extension of the XXZ model, free field realization at level $k > 1$ is required, which was constructed by several authors, such as [16, 17, 18]. Furthermore, in [19], Wakimoto representation of $U_q(\widehat{sl_N})$ with arbitrary level $k \geq 1$ was given, and it plays a central role in understanding higher rank extension of the XXZ model. Free field method is also a powerful way to study the integrable massive field theory [20]. Please see [21] for a nice review on quantum affine algebra, free field realization and their applications. The level k free field representation of Yangian double $DY_\hbar(sl_2)$ and application to physical problems were discussed in [22, 23]. The level one free field realization of the Yangian double with central $DY_\hbar(sl_N)$ was constructed in [24]; while the level k representation of $DY_\hbar(sl_N)$ and $DY_\hbar(gl_N)$ were given in [25]. It should also be remarked that the Yangian double with central $DY_\hbar(\widehat{sl_2})$ is the symmetry possessed by the Sine-Gordon model, which is the field theory limit of the restricted SOS (RSOS) model [26, 27]. The bosonization of the RSOS model was considered in [28].

So following the algebraic approach, it is also important to obtain the free fields realization of elliptic quantum algebras. For example, in studying the RSOS model and its higher spin extension (i.e. the k-fusion RSOS model), the free field representation of $U_{q,p}(\widehat{sl_2})$ with any given level k is needed, which has been presented in [29] and this construction corresponds to a deformation of special coset WZW model. In fact, the elliptic algebra $U_{q,p}(\widehat{sl_2})$ is the Drinfeld realization of $B_{q,\lambda}(\widehat{sl_2})$ showed in [30].

In fact, at classical level, there are various models of representation for the current algebra, each one has its significance in certain application. Here we just mention two of them: free boson representation (Wakimoto construction) [9, 10], and parafermion realization [32, 33, 34]. The similar realizations have been extended to quantum affine algebras and Yangian double with central. But there is no similar extension for the elliptic quantum algebras, except the parafermion realization for $U_{q,p}(\widehat{sl_2})$ of higher level k [29], and the parafermion representation of $U_{q,p}(\widehat{sl_N})$ with level one [31]. In fact, the realization presented in [29] is obtained by twisting the parafermionic realization of the quantum affine algebra $U_q(\widehat{sl_2})$, which can be considered as the elliptic deformation of the parafermion realization. The $su(2)$ parafermion currents can be identified with the coset WZW model of $\widehat{sl(2)}_k \otimes \widehat{sl(2)}_1 / \widehat{sl(2)}_{k+1}$. Although parafermion theory is important in physics [32, 33, 34] and in mathematics [35], it seems that it cannot be used directly to study the elliptic quantum algebra. In fact, the bosonization of non-local currents for higher rank algebras is a huge project even in the classical level. So if one want to deal with the elliptic quantum algebra of higher rank, through bosonization of the non-local currents is not a practical way.

It has special interest for the algebra of the intertwining operators in the WZW model. It is derived by Knizhnik and Zamolodchikov that the matrix coefficients of the intertwining operators for the WZW model satisfy certain holonomic differential equations, i.e., the Knizhnik-Zamolodchikov (KZ) equation [36]. There is an analogue holonomic q-difference equation for the quantum affine vertex operators. They satisfy the quantum (q-deformed) Knizhnik-Zamolodchikov (q-KZ) equation [37]. So it is also expected that the representations of the elliptic quantum algebras are likely to be helpful to construct the elliptic type solutions of quantum Knizhnik-Zamolodchikov-Bernard (q-KZB) equation, which is a higher genus extension of the q-KZ equation [38]. Furthermore, to consider a higher rank extension of the RSOS model, we should construct free boson realization of $U_{q,p}(\widehat{sl_N})$. However only in the level-one case, parafermion realization of it was given in [31], free boson realization of it with higher level is unknown at present.

In this paper, we present a new free boson representation of $U_{q,p}(\widehat{sl_2})$ with arbitrary level k. It is different from the known one in [29] which was constructed in terms of non-local currents. Our construction could be viewed as a twisted version of the quantum semi-infinite flag manifolds [10], which is called the elliptic version of Wakimoto realization. The realization of the quantum intertwining operators, such as the screening currents and the vertex operators are also given. They are necessary ingredients for calculating correlation functions and investigating the irreducible representations. The screening currents commute or anti-commute with $U_{q,p}(\widehat{sl_2})$, and the integrations of such currents give the screening charges. For $U_{q,p}(\widehat{sl_2})$, there are two types of intertwining operators, which are called the type *I* vertex operators (VO) and the type *II* vertex operators respectively with their different physical significance. The former is a local operator which describes the operation of adding one lattice site, and the formula of the correlation functions can be expressed as traces of the product of these operators over irreducible representation space; while the latter play the role of particle creation or annihilation operators. In fact, in this paper we construct two screening currents, and the two types of VO's as well as their twisted ones. Moreover we hope this construction can be generalized to other cases, which will be considered in future study [39].

The paper is organized as follows. In section 2 we fix notations and recall the definition of $U_{q,p}(\widehat{sl_2})$. We give our construction of the free boson realization in section 3. In section 4, two screening currents of it are constructed in terms of free bosons. Finally in section 5 we give the free boson realization of the type *I* VO's, the type *II* VO's and their twisted ones.

2 The definition of elliptic algebra $U_{q,p}(\widehat{sl_2})$

Elliptic quantum algebras are introduced to study integrable models with elliptic Boltzmann weights. There are two types of them: the vertex type and the face type. Here we restrict to the face type $B_{q,\lambda}(\mathcal{G})$ with $\mathcal{G} = \widehat{sl_2}$. For the face type elliptic quantum algebras $U_{q,p}(\widehat{sl_2})$, it can be considered as the Drinfeld realization of $B_{q,\lambda}(\widehat{sl_2})$. In this section, we give a short review on its definition.

Let us introduce a pair of parameters p and p^*:

$$p = q^{2r}, \quad p^* = q^{2r^*} = pq^{-2c} \qquad (r^* = r - c; \ \ r, r^* \in \mathbb{R}_{>0})$$

here c is the central element of the elliptic algebra $U_{q,p}(\widehat{sl_2})$ defined below. Throughout this paper, the complex number $q \neq 0$, $|q| < 1$ is fixed.

Definition 1 . *The associative algebra $U_{q,p}(\widehat{sl_2})$ is generated by the central element c and the operator-valued currents $H^\pm(z)$, $E(z)$ and $F(z)$ of the complex variable z satisfying the following commutation relations:*

$$H^\pm(z)H^\pm(w) = \left(\frac{z}{w}\right)^{2\left(\frac{1}{r^*}-\frac{1}{r}\right)}\frac{\Theta_p(q^{-2}\frac{z}{w})}{\Theta_p(q^2\frac{z}{w})}\frac{\Theta_{p^*}(q^2\frac{z}{w})}{\Theta_{p^*}(q^{-2}\frac{z}{w})}H^\pm(w)H^\pm(z), \tag{2.1}$$

$$H^+(z)H^-(w) = q^{2c\left(\frac{1}{r^*}+\frac{1}{r}\right)}\left(\frac{z}{w}\right)^{2\left(\frac{1}{r^*}-\frac{1}{r}\right)}\frac{\Theta_p(pq^{-2-c}\frac{z}{w})}{\Theta_p(pq^{2-c}\frac{z}{w})}\frac{\Theta_{p^*}(p^*q^{2+c}\frac{z}{w})}{\Theta_{p^*}(p^*q^{-2+c}\frac{z}{w})}H^-(w)H^+(z), \tag{2.2}$$

$$H^\pm(z)E(w) = q^{\pm\frac{c}{r^*}-2}\left(\frac{z}{w}\right)^{\frac{2}{r^*}}\frac{\Theta_{p^*}(q^{2\pm\frac{c}{2}}\frac{z}{w})}{\Theta_{p^*}(q^{-2\pm\frac{c}{2}}\frac{z}{w})}E(w)H^\pm(z), \tag{2.3}$$

$$H^\pm(z)F(w) = q^{\pm\frac{c}{r}+2}\left(\frac{z}{w}\right)^{-\frac{2}{r}}\frac{\Theta_p(q^{-2\mp\frac{c}{2}}\frac{z}{w})}{\Theta_p(q^{2\mp\frac{c}{2}}\frac{z}{w})}F(w)H^\pm(z), \tag{2.4}$$

$$[E(z), F(w)] = \frac{1}{(q-q^{-1})zw}[\delta(q^{-c}\frac{z}{w})H^+(q^{-\frac{c}{2}}z) - \delta(q^c\frac{z}{w})H^-(q^{-\frac{c}{2}}w)], \tag{2.5}$$

$$E(z)E(w) = q^{-2}\left(\frac{z}{w}\right)^{\frac{2}{r^*}}\frac{\Theta_{p^*}(q^2\frac{z}{w})}{\Theta_{p^*}(q^{-2}\frac{z}{w})}E(w)E(z), \tag{2.6}$$

$$F(z)F(w) = q^2\left(\frac{z}{w}\right)^{-\frac{2}{r}}\frac{\Theta_p(q^{-2}\frac{z}{w})}{\Theta_p(q^2\frac{z}{w})}F(w)F(z), \tag{2.7}$$

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where

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n,$$

$$\Theta_p(z) = (z; p)_\infty (pz^{-1}; p)_\infty (p; p)_\infty,$$

$$(z; t_1, \cdots, t_k)_\infty = \prod_{n_1, \cdots, n_k \geq 0} (1 - z t_1^{n_1} \cdots t_k^{n_k}),$$

by the definition, $\Theta_p(z)$ is the standard elliptic theta-function, up to a constant. In general, we can define $\Theta_t(z)$ for any parameter $t = q^{2\nu}$ ($\nu \in \mathbb{C}$) as

$$\Theta_t(z) = (z; t)_\infty (tz^{-1}; t)_\infty (t; t)_\infty.$$

It should also be remarked that this elliptic algebra $U_{q,p}(\widehat{sl_2})$ degenerates to the quantum affine algebra $U_q(\widehat{sl_2})$ in the $p \to 0$ (or $r \to \infty$) limit.

In order to rewrite the relations (2.1)-(2.7) in more elegant form, the following parameterization will be used:

$$q = e^{-\pi i/r\tau},$$

$$p = e^{-2\pi i/\tau}, \quad p^* = e^{-2\pi i/\tau^*}$$

$$z = q^{2u} = e^{-2\pi i u/r\tau}.$$

In fact, we can use the notation of Jacobi theta function

$$\theta_\nu(u) = q^{\frac{u^2}{\nu} - u} \frac{\Theta_{q^{2\nu}}(q^{2u})}{(q^{2\nu}; q^{2\nu})_\infty^3};$$

however for simplicity, we denote $\theta_r(u)$ as $\theta(u)$ and $\theta_{r^*}(u)$ as $\theta^*(u)$, which satisfy the quasi-periodicity properties

$$\theta(u + r) = -\theta(u), \qquad \theta(u + r\tau) = -e^{-\pi\tau i - 2\pi i u/r} \theta(u),$$

and similar relations hold for $\theta^*(u)$ with r replaced by r^*. Then it is obvious to see that (2.1)-(2.7) can be rewritten as follows:

$$H^\pm(u)H^\pm(v) = \frac{\theta(u - v - 1)}{\theta(u - v + 1)} \frac{\theta^*(u - v + 1)}{\theta^*(u - v - 1)} H^\pm(v)H^\pm(u), \tag{2.8}$$

$$H^+(u)H^-(v) = \frac{\theta(u - v - c/2 - 1)}{\theta(u - v - c/2 + 1)} \frac{\theta^*(u - v + c/2 + 1)}{\theta^*(u - v + c/2 - 1)} H^-(v)H^+(u), \tag{2.9}$$

$$H^\pm(u)E(v) = \frac{\theta^*(u - v \pm c/4 + 1)}{\theta^*(u - v \pm c/4 - 1)} E(v)H^\pm(u), \tag{2.10}$$

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$$H^{\pm}(u)F(v) = \frac{\theta(u - v \mp c/4 - 1)}{\theta(u - v \mp c/4 + 1)}F(v)H^{\pm}(u), \tag{2.11}$$

$$[E(u), F(v)] = \frac{1}{(q - q^{-1})zw}[\delta(u - v - c/2)H^{+}(u - c/4)$$

$$-\delta(u - v + c/2)H^{-}(v - c/4)], \tag{2.12}$$

$$E(u)E(v) = \frac{\theta^{*}(u - v + 1)}{\theta^{*}(u - v - 1)}E(v)E(u), \tag{2.13}$$

$$F(u)F(v) = \frac{\theta(u - v - 1)}{\theta(u - v + 1)}F(v)F(u), \tag{2.14}$$

where the parameterizations $z = q^{2u}$ and $w = q^{2v}$ are implicit in the above expressions. In the following, we will use this parameterization without mentioning them if they do not make confusion. Note that the above exchange relations have nice periodicity property with the notations of the Jacobi theta functions.

3 Fock Realization of $U_{q,p}(\widehat{sl_2})$ currents

In this section, we construct an elliptic deformation of Wakimoto realization for $U_{q,p}(\widehat{sl_2})$ using the free fields representation of $U_q(\widehat{sl_2})$ for generic level k. We first fix some conventions and review the Wakimoto realization of $U_q(\widehat{sl_2})$; then give our construction of the realization for $U_{q,p}(\widehat{sl_2})$ currents.

3.1 Fock space and the quantum affine algebra $U_q(\widehat{sl_2})$

Three kinds free bosons a, b and c are needed to construct the realization of $U_q(\widehat{sl_2})$. Their commutation relations of modes are

$$[a_n, a_m] = \frac{[(k + 2)n][2n]}{n}\delta_{n+m,0}, \quad [p_a, q_a] = 2(k + 2),$$

$$[b_n, b_m] = -\frac{[n]^2}{n}\delta_{n+m,0}, \quad [p_b, q_b] = -1,$$

$$[c_n, c_m] = \frac{[n]^2}{n}\delta_{n+m,0}, \quad [p_c, q_c] = 1,$$

and the others vanish, where k is generic with $k \neq -2$. Throughout this paper, the following standard symbol $[n]$ will be used: $[n] = (q^n - q^{-n})/(q - q^{-1})$.

The vacuum state of the Fock space $|\mathbf{0}\rangle \equiv |0, 0, 0\rangle$ is set as

$$a_n|\mathbf{0}\rangle = b_n|\mathbf{0}\rangle = c_n|\mathbf{0}\rangle = 0 \quad (n \geq 0),$$

and a state $|l, m_1, m_2\rangle$ is produced through

$$|l, m_1, m_2\rangle \equiv \exp\{l\ q_a/2(k+2) + m_1 q_b + m_2 q_c\}|\mathbf{0}\rangle.$$

Obviously $|l, m_1, m_2\rangle$ is the highest weight state of the bosonic Fock space, which is uniquely characterized by:

$$a_n|l, m_1, m_2\rangle = b_n|l, m_1, m_2\rangle = c_n|l, m_1, m_2\rangle = 0 \quad (n > 0),$$
$$p_a|l, m_1, m_2\rangle = l|l, m_1, m_2\rangle, \quad p_b|l, m_1, m_2\rangle = -m_1|l, m_1, m_2\rangle,$$
$$p_c|l, m_1, m_2\rangle = m_2|l, m_1, m_2\rangle,$$

then the Fock space $\mathcal{F}_{l,m_1,m_2}$ is generated by negative modes a_n, b_n and c_n $(n < 0)$ acting on the highest weight state $|l, m_1, m_2\rangle$. The dual Fock space could be constructed with the same matter.

For convenience, we denote free boson fields $a(z; \alpha)$ with $\alpha \in \mathbb{C}$ and $a_\pm(z)$ as follows:

$$a(z; \alpha) = -\sum_{n \neq 0} \frac{a_n}{[n]} q^{-\alpha|n|} z^{-n} + q_a + p_a \ln z,$$
$$a_\pm(z) = \pm((q - q^{-1}) \sum_{n > 0} a_{\pm n} z^{\mp n} + p_a \ln q),$$

and $a(z; 0) \equiv a(z)$ for simplicity. Similarly, the free boson fields $b(z; \alpha)$, $b_\pm(z)$ and $c(z; \alpha)$, $c_\pm(z)$ can also be given. Normal order prescription $: :$ is set by moving $a_n(n > 0)$ and p_a to right, while moving $a_n(n < 0)$ and q_a to left. For example,

$$: \exp(a(z)) := \exp(-\sum_{n < 0} \frac{a_n}{[n]} z^{-n}) e^{q_a} z^{p_a} \exp(-\sum_{n > 0} \frac{a_n}{[n]} z^{-n}).$$

With the help of the above free bosonic fields, four fields $\psi_\pm(z)$ and $e^\pm(z)$ are introduced through their actions on the Fock space $\mathcal{F}_{l,m_1,m_2}$. Let us fix the actions of these currents on the Fock space as: $\psi_\pm(z)$: $\mathcal{F}_{l,m_1,m_2} \mapsto \mathcal{F}_{l,m_1,m_2}$, $e^+(z) : \mathcal{F}_{l,m_1,m_2} \mapsto \mathcal{F}_{l,m_1-1,m_2-1}$ and $e^-(z) : \mathcal{F}_{l,m_1,m_2} \mapsto \mathcal{F}_{l,m_1+1,m_2+1}$, respectively. Then these currents can be expressed as follows:

$$\psi_+(z) =: \exp[b_+(q^{\frac{k}{2}} z) + a_+(qz) + b_+(q^{\frac{k}{2}+2} z)] :,$$
$$\psi_-(z) =: \exp[b_-(q^{-\frac{k}{2}} z) + a_-(q^{-1} z) + b_-(q^{-(\frac{k}{2}+2)} z)] :,$$

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$$e^+(z) = \frac{-1}{(q-q^{-1})z} : \{\exp[b_+(z) - (b+c)(qz)] - \exp[b_-(z) - (b+c)(q^{-1}z)]\} :,$$

$$e^-(z) = \frac{-1}{(q-q^{-1})z} : \{\exp[(b+c)(q^{-(k+1)}z)] \exp[a_-(q^{-(\frac{k+2}{2})}z) + b_-(q^{-(k+2)}z)]$$

$$- \exp[(b+c)(q^{k+1}z)] \exp[a_+(q^{\frac{k+2}{2}}z) + b_+(q^{k+2}z)]\} :;$$

and the following proposition is straightforward:

Proposition 1 . *The fields given above satisfy the following commutation relations [19]:*

$$[\psi_\pm(z), \psi_\pm(w)] = 0, \tag{3.1}$$

$$(z - q^{2-k}w)(z - q^{-2+k}w)\psi_+(z)\psi_-(w)$$

$$= (z - q^{2+k}w)(z - q^{-2-k}w)\psi_-(w)\psi_+(z), \tag{3.2}$$

$$(z - q^{\pm(2-\frac{k}{2})}w)\psi_+(z)e^\pm(w) = (q^{\pm2}z - q^{\mp\frac{k}{2}}w)e^\pm(w)\psi_+(z), \tag{3.3}$$

$$(z - q^{\pm(2-\frac{k}{2})}w)e^\pm(z)\psi_-(w) = (q^{\pm2}z - q^{\mp\frac{k}{2}}w)\psi_-(w)e^\pm(z), \tag{3.4}$$

$$[e^+(z), e^-(w)] = \frac{1}{(q-q^{-1})zw}[\delta(q^{-k}\frac{z}{w})\psi_+(q^{-\frac{k}{2}}z) - \delta(q^k\frac{z}{w})\psi_-(q^{-\frac{k}{2}}w)], \tag{3.5}$$

$$(z - q^{\pm2}w)e^\pm(z)e^\pm(w) = (q^{\pm2}z - w)e^\pm(w)e^\pm(z). \tag{3.6}$$

As a result, the above bosonic expression of the fields $\psi_\pm(z)$ and $e^\pm(z)$ give the Wakimoto realization of the quantum affine algebra $U_q(\widehat{sl_2})$ for generic level k.

3.2 Free boson realization of $U_{q,p}(\widehat{sl_2})$

In this subsection, we will present a new free fields realization of $U_{q,p}(\widehat{sl_2})$ with given level k. The elliptic algebra $U_{q,p}(\widehat{sl_2})$ can be realized as the tensor product of the elliptic currents $\Psi^\pm(z)$, $e(z)$, $f(z)$ of $U_q(\widehat{sl_2})$ and a Heisenberg algebra [30]. The elliptic currents $\Psi^\pm(z)$, $e(z)$ and $f(z)$ of $U_q(\widehat{sl_2})$ are the fields satisfying the following elliptic commutation relations:

$$\Psi^\pm(z)\Psi^\pm(w) = \frac{\Theta_p(q^{-2}\frac{z}{w})}{\Theta_p(q^2\frac{z}{w})} \frac{\Theta_{p^*}(q^2\frac{z}{w})}{\Theta_{p^*}(q^{-2}\frac{z}{w})} \Psi^\pm(w)\Psi^\pm(z), \tag{3.7}$$

$$\Psi^+(z)\Psi^-(w) = \frac{\Theta_p(pq^{-2-c}\frac{z}{w})}{\Theta_p(pq^{2-c}\frac{z}{w})} \frac{\Theta_{p^*}(p^*q^{2+c}\frac{z}{w})}{\Theta_{p^*}(p^*q^{-2+c}\frac{z}{w})} \Psi^-(w)\Psi^+(z), \tag{3.8}$$

$$\Psi^{\pm}(z)e(w) = q^{-2}\frac{\Theta_{p^*}(q^{2\pm\frac{c}{2}}\frac{z}{w})}{\Theta_{p^*}(q^{-2\pm\frac{c}{2}}\frac{z}{w})}e(w)\Psi^{\pm}(z),$$

(3.9)

$$\Psi^{\pm}(z)f(w) = q^{2}\frac{\Theta_{p}(q^{-2\mp\frac{c}{2}}\frac{z}{w})}{\Theta_{p}(q^{2\mp\frac{c}{2}}\frac{z}{w})}f(w)\Psi^{\pm}(z),$$

(3.10)

$$[e(z), f(w)] = \frac{1}{(q-q^{-1})zw}[\delta(q^{-c}\frac{z}{w})\Psi^{+}(q^{-\frac{c}{2}}z) - \delta(q^{c}\frac{z}{w})\Psi^{-}(q^{-\frac{c}{2}}w)],$$

(3.11)

$$e(z)e(w) = q^{-2}\frac{\Theta_{p^*}(q^{2}\frac{z}{w})}{\Theta_{p^*}(q^{-2}\frac{z}{w})}e(w)e(z),$$

(3.12)

$$f(z)f(w) = q^{2}\frac{\Theta_{p}(q^{-2}\frac{z}{w})}{\Theta_{p}(q^{2}\frac{z}{w})}f(w)f(z).$$

(3.13)

In [30] the elliptic currents $\Psi^{\pm}(z)$, $e(z)$ and $f(z)$ of $U_q(\widehat{sl_2})$ are realized by twisted the parafermion realization of quantum affine algebra $U_q(\widehat{sl_2})$. In the following, we will give another bosonic realization, which can be viewed as the twisted quantum version of the realization on flag manifold given by B. Feigin and E. Frenkel [10]. To get the bosonic representation of these elliptic currents, besides the bosonic fields introduced in the last subsection, we need some new ones, such as $a^*_{\pm}(z)$,

$$a^*_+(z) = -\sum_{n>0}\frac{a_n}{[rn]}z^{-n},$$

$$a^*_-(z) = \sum_{n>0}\frac{a_{-n}}{[r^*n]}z^{n}.$$

Here we name them as the twisted partners of the fields $a_{\pm}(z)$ respectively; and $b^*_{\pm}(z)$ are introduced with the same matter by replacing $a_{\pm n}$ with $b_{\pm n}$ in the above expressions. In terms of them we introduce two twisting currents $U^{\pm}(z; r, r^*)$ depending on parameters r and r^* as:

$$U^+(z; r, r^*) = \exp[a^*_-(q^{r^*+\frac{k}{2}-1}z) + b^*_-(q^{r^*-1}(q + q^{-1})z)],$$

$$U^-(z; r, r^*) = \exp[a^*_+(q^{-(r-\frac{k}{2}-1)}z) + b^*_+(q^{-(r-k-1)}(q + q^{-1})z)].$$

By twisting the fields $\psi_{\pm}(z)$ and $e^{\pm}(z)$ in subsection 3.1 with $U^{\pm}(z; r, r^*)$, we obtain the fields $\Psi^{\pm}(z)$, $e(z)$ and $f(z)$ as:

$$\Psi^+(z) = U^+(q^{\frac{k}{2}}z; r, r^*)\psi_+(z)U^-(q^{-\frac{k}{2}}z; r, r^*),$$

(3.14)

$$\Psi^-(z) = U^+(q^{-\frac{k}{2}}z; r, r^*)\psi_-(z)U^-(q^{\frac{k}{2}}z; r, r^*),$$

(3.15)

$$e(z) = U^+(z; r, r^*)e^+(z),$$

(3.16)

$$f(z) = e^-(z)U^-(z; r, r^*).$$

(3.17)

Obviously, the actions of the fields $\Psi^\pm(z)$, $e(z)$ and $f(z)$ on the Fock space $\mathcal{F}_{l,m_1,m_2}$ are the same as $\psi_\pm(z)$, $e^+(z)$ and $e^-(z)$ respectively, and we have the following proposition:

Proposition 2 . *The fields $\Psi^\pm(z)$, $e(z)$ and $f(z)$ obtained above with $k = c$ satisfy the given elliptic commutation relations (3.7)-(3.13).*

Proof: A straightforward but length OPE calculation verifies this proposition. Here we just list some useful formulas:

$$e^A e^B = e^{[A,B]} e^B e^A, \quad if\ [A,B]\ commute\ with\ A\ and\ B;$$

$$\exp(-\sum_{n>0} \frac{x^n}{n}) = 1 - x;$$

$$(1 - x)^{-1} = \sum_{n\geq 0} x^n. \qquad \square$$

By this proposition, we state that the currents in (3.14)-(3.17) with $k = c$ give a bosonization of the elliptic currents of $U_q(\widehat{sl_2})$. From their actions on the Fock space, we know that all the currents keep the "spin-$l/2$" representation. Furthermore, it should also be remarked that, in the $p \to 0$ limit, $\Psi^+(z) \to (\psi_-(q^k z))^{-1}$, $\Psi^-(z) \to (\psi_+(q^k z))^{-1}$, $e(z) \to q^{-(2p_b+p_a)}(\psi_-(q^{k/2}z))^{-1}e^+(z)$ and $f(z) \to e^-(z)q^{2p_b+p_a}(\psi_+(q^{k/2}z))^{-1}$ give a new free fields realization of $U_q(\widehat{sl_2})$. It is different from the one given in subsection 3.1.

However, the exchange relations of the currents given by the above boson fields do not have good periodicity property. In order to touch that goal, i.e., to construct a free boson realization of the elliptic quantum algebra $U_{q,p}(\widehat{sl_2})$, we need introduce a Heisenberg algebra generated by $\hat{p}$ and $\hat{q}$ such that

$$[\hat{q}, \hat{p}] = 1,$$

and they commute with a, b and c. With the help of them we have new fields

$$H^\pm(u) = \Psi^\pm(z)e^{2\hat{q}}(q^{\mp\frac{k}{2}}z)^{\frac{2p_b+p_a}{r}}(q^{\pm(r-\frac{k}{2})}z)^{\frac{\hat{p}-1}{r}-\frac{\hat{p}-1}{r^*}}, \tag{3.18}$$

$$E(u) = e(z)e^{2\hat{q}}z^{-\frac{\hat{p}-1}{r^*}}, \tag{3.19}$$

$$F(u) = f(z)z^{\frac{2p_b+p_a}{r}}z^{\frac{\hat{p}-1}{r}}. \tag{3.20}$$

To see them more clearly, we turn to the Fock space structure. From the above expressions, the Fock space structure of $H^\pm(u)$, $E(u)$ and $F(u)$ could be given as tensor product of Fock spaces of $\Psi^\pm(z)$, $e(z)$ and $f(z)$ respectively, with certain ones generated by $\hat{q}$. The results of their actions are $H^\pm(u) : \mathcal{F}_{l,m_1,m_2} \otimes \mathcal{F}_n \mapsto \mathcal{F}_{l,m_1,m_2} \otimes \mathcal{F}_{n+2}$, $E(u) : \mathcal{F}_{l,m_1,m_2} \otimes \mathcal{F}_n \mapsto \mathcal{F}_{l,m_1-1,m_2-1} \otimes \mathcal{F}_{n+2}$ and

$F(u) : \mathcal{F}_{l,m_1,m_2} \otimes \mathcal{F}_n \mapsto \mathcal{F}_{l,m_1+1,m_2+1} \otimes \mathcal{F}_n$. Here $\mathcal{F}_n$ is a trivial Fock space generated by $|n\rangle \equiv e^{n\hat{q}}|0\rangle$. Then by applying proposition 2 and direct calculation, we can verify the following theorem:

Theorem 1 . *The fields in Eqns. (3.18)-(3.20) with $k = c$ satisfy the commutation relations (2.8)-(2.14).*

Corollary 1 . *$H^{\pm}(u)$, $E(u)$ and $F(u)$ given above realize the elliptic algebra $U_{q,p}(\widehat{sl_2})$ with given level $k = c$.*

4   Construction of the screening currents

In 2D CFT, screening current is a primary field of the energy-momentum tensor with conformal weight 1, and its integration gives the screening charge. It has the property that it commutes with the currents modulo a total differential of certain field. This property ensures that the screening charge may be inserted in the correlators by changing their conformal charges without affecting their conformal properties. In this section, using the bosons a, b and c, we construct two screening currents $S_I(z)$ and $S_{II}(z)$, which are integral parts in the free fields approach. The two currents in this section commute or anti-commute with the currents modulo a total q-difference of some fields, so they could be regarded as a quantum deformation of the screening currents in 2D CFT.

Denote a sort of q-difference operator with a parameter $n \in \mathbb{Z}_{>0}$ by

$$_n\partial_z X(z) \equiv \frac{X(q^n z) - X(q^{-n} z)}{(q - q^{-1})z},$$

which is called a total q-difference of a function $X(z)$. Moreover to eliminate the total q-difference, one can define the Jackson integral as

$$\int_0^{s\infty} X(z) d_p z \equiv s(1 - p) \sum_{n \in \mathbb{Z}} X(sp^n) p^n,$$

for a scalar $s \in \mathbb{C}\backslash\{0\}$ and a complex number p such that $| p |< 1$. So that,

$$\int_0^{s\infty} (_n\partial_z X(z)) d_p z = 0,$$

if it is convergent and take $p = q^{2n}$. For simplicity, we denote boson fields with parameters L_i and M_j ($i, j \in \mathbb{N}$) as follows:

$$A_+(L_1, \cdots, L_s; M_1, \cdots, M_{s+1}|z; \alpha) = \sum_{n>0} \frac{[L_1 n] \cdots [L_s n]}{[M_1 n] \cdots [M_{s+1} n]} a_n (q^\alpha z)^{-n},$$

$$A_-(L_1, \cdots, L_s; M_1, \cdots, M_{s+1}|z; \alpha) = \sum_{n>0} \frac{[L_1 n] \cdots [L_s n]}{[M_1 n] \cdots [M_{s+1} n]} a_{-n} (q^\alpha z)^{n},$$

and further abbreviate the notations as:

$$A_\pm(L_1, \cdots, L_s; M_1, \cdots, M_{s+1}|z) = A_\pm(L_1, \cdots, L_s; M_1, \cdots, M_{s+1}|z; 0),$$

$$A_\pm(M|z; \alpha) = A_\pm(L_1, \cdots, L_s; L_1, \cdots, L_s, M|z; \alpha);$$

similarly the fields $B_\pm(L_1, \cdots, L_s; M_1, \cdots, M_{s+1}|z; \alpha)$ and $C_\pm(L_1, \cdots, L_s; M_1, \cdots, M_{s+1}|z; \alpha)$ can also be given.

Using these fields we obtain the screening currents as:

$$S_I(z) =: \exp\{c(z) + \frac{q_a}{2} + q_b - r^* \hat{q}\} :,$$

$$S_{II}(z) = \frac{-1}{(q - q^{-1})z} : \exp\{A_+(k + 2|z; \frac{k+2}{2}) - \frac{1}{k+2}(q_a + p_a \ln z)$$

$$+ A_-(-(k+2)|z; -\frac{k+2}{2})\}$$

$$\times \{\exp[-b_-(z) - (b+c)(qz)] - \exp[-b_+(z) - (b+c)(q^{-1}z)]\} :,$$

since they have the following properties:

Theorem 2 : $S_I(z)$, $S_{II}(z)$ *satisfy the following relations with the currents $H^\pm(z)$, $E(z)$ and $F(z)$ given by (3.18)-(3.20):*

$$H^\pm(z) S_I(w) = S_I(w) H^\pm(z) = O(1),$$

$$E(z) S_I(w) = -S_I(w) E(z) = {}_1 \partial_w [\frac{1}{z-w} \tilde{s}_1(z)] + O(1),$$

$$F(z) S_I(w) = -S_I(w) F(z) = O(1),$$

$$S_I(z) S_I(w) = -S_I(w) S_I(z) = O(1);$$

$$H^\pm(z) S_{II}(w) = S_{II}(w) H^\pm(z) = O(1),$$

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$$E(z)S_{II}(w) = S_{II}(w)E(z) = O(1),$$

$$F(z)S_{II}(w) = S_{II}(w)F(z) = {}_{(k+2)}\partial_w[\frac{1}{z-w}\tilde{s}_2(z)] + O(1),$$

$$S_{II}(z)S_{II}(w) = \frac{\theta_{k+2}(u-v+1)}{\theta_{k+2}(u-v-1)}S_{II}(w)S_{II}(z);$$

$$S_{II}(z)S_I(w) = -S_I(w)S_{II}(z) = {}_1\partial_w[\frac{1}{z-w}\tilde{s}_3(z)] + O(1),$$

where the symbol $O(1)$ means regularity and $\tilde{s}_i(z)$ $(i = 1, 2, 3)$ are given by:

$$\tilde{s}_1(z) =: \exp\{A_-(r^*|z; r - \frac{k}{2} - 1) + \frac{q_a}{2} + B_-(-(r-k-2); r^*, 1|z; -1)$$

$$-p_b \ln z + B_+(1|z; -1) + (2 - r^*)\hat{q} - \frac{\hat{p}-1}{r^*}\ln z\} :,$$

$$\tilde{s}_2(z) =: \exp\{A_-(-(k+2)|z; \frac{k+2}{2}) + A_+(r-k-2; k+2, r|z; \frac{k+2}{2})$$

$$-\frac{1}{k+2}(q_a + p_a \ln z) - B_+(2; 1, r|z; -(r-k-1)) + \frac{2p_b + p_a + \hat{p} - 1}{r}\ln z\} :,$$

$$\tilde{s}_3(z) =: \exp\{A_-(-(k+2)|z; -\frac{k+2}{2}) + A_+(k+2|z; \frac{k+2}{2})$$

$$+\frac{1}{2(k+2)}(kq_a - 2p_a \ln z) - b(z; 1) + q_b - r^*\hat{q}\} : .$$

Proof: Straightforward calculation. Here we only take the last relation as an example, we denote

$$S_{II}(z) \equiv \frac{-1}{(q-q^{-1})z}[A(z) - B(z)]$$

where

$$A(z) =: \exp\{A_-(-(k+2)|z; -\frac{k+2}{2}) + A_+(k+2|z; \frac{k+2}{2})$$

$$-\frac{1}{k+2}(q_a + p_a \ln z) - b_-(z) - (b+c)(qz)\} :,$$

$$B(z) =: \exp\{A_-(-(k+2)|z; -\frac{k+2}{2}) + A_+(k+2|z; \frac{k+2}{2})$$

$$-\frac{1}{k+2}(q_a + p_a \ln z) - b_+(z) - (b+c)(q^{-1}z)\} :,$$

then

$$S_{II}(z)S_I(w) = \frac{-1}{(q-q^{-1})z}[A(z)S_I(w) - B(z)S_I(w)],$$

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and since the following relations hold:

$$A(z)S_I(w) = \frac{1}{qz - w} : A(z)S_I(w) :, \quad |z| > |w|;$$

$$S_I(w)A(z) = \frac{1}{w - qz} : A(z)S_I(w) :, \quad |w| > |z|;$$

$$B(z)S_I(w) = \frac{1}{q^{-1}z - w} : B(z)S_I(w) :, \quad |z| > |w|;$$

$$S_I(w)B(z) = \frac{1}{w - q^{-1}z} : B(z)S_I(w) :, \quad |w| > |z|,$$

we obtain the following relation on the analytic continuations:

$$S_{II}(z)S_I(w) = -S_I(w)S_{II}(z) = \frac{-1}{(q - q^{-1})z}[\frac{1}{qz - w} : A(z)S_I(w) : -\frac{1}{q^{-1}z - w} : B(z)S_I(w) :];$$

moreover,

$$: A(z)S_I(qz) :=: B(z)S_I(q^{-1}z) :\equiv \tilde{s}_3(z);$$

then by the definition of the total q-difference given above, we get

$$S_{II}(z)S_I(w) = -S_I(w)S_{II}(z) = {}_1\partial_w[\frac{1}{z - w}\tilde{s}_3(z)] + O(1). \quad \square$$

It is easy to see that the actions of screening currents on the Fock space are $S_I(z) : \mathcal{F}_{l,m_1,m_2} \otimes \mathcal{F}_n \mapsto \mathcal{F}_{l+(k+2),m_1+1,m_2+1} \otimes \mathcal{F}_{n-r^*}$ and $S_{II}(z) : \mathcal{F}_{l,m_1,m_2} \otimes \mathcal{F}_n \mapsto \mathcal{F}_{l-2,m_1-1,m_2-1} \otimes \mathcal{F}_n$, respectively. Please note that the current $S_{II}(z)$ acts trivially on $\mathcal{F}_n$, so it is also the screening current of the elliptic currents $\Psi^{\pm}(z)$, $e(z)$ and $f(z)$. On the other side, the current $S_I(z)$ is not screening operator of them, even if we remove the term involving $\hat{q}$ in $S_I(z)$ by hand. In fact, using the expressions of $S_I(z)$ and $S_{II}(z)$ given above, we can calculate the cohomology of the algebra and study the irreducibility of modules of it, which will be discussed separately in the future.

5 Realization of the Vertex Operators

In fact, in 2D CFT, besides the screening currents, the other important object that one should discuss is the primary field. In WZW model, the primary fields could be realized as the highest weight representation of Kac-Moody algebra, which are commonly known as vertex operators (VOs)or intertwiner operators. They play crucial role in calculating correlation functions. For quantum affine algebra, there are two types of vertex operators [21] or intertwiner operators, in which the type I is a local operator and could be regarded as the quantum counterpart of the

primary field in 2D CFT. In this section, we'll present a new realization of the two types vertex operators and their twisted ones, which are different from the ones given in [29], as they base on distinct free fields realization of $U_{q,p}(\widehat{sl_2})$. For the definitions of the VO's and the properties of them, please see [21, 30] for more details.

5.1 The type I and the type II Vertex Operators

For $U_{q,p}(\widehat{sl_2})$, the type I vertex operators and the type II vertex operators are defined to be the operators:

$$\widehat{\Phi}_l(u) : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}} \otimes V_{l,v} \tag{5.1}$$

$$\widehat{\Psi}_l^*(u) : V_{l,v} \otimes \widehat{\mathcal{F}} \to \widehat{\mathcal{F}} \tag{5.2}$$

acting on the total Fock space $\widehat{\mathcal{F}}$, where $V_{l,v}$ is the spin $\frac{l}{2}$ representation generated by vectors v_m^l ($m = 0, \cdots, l$). For convenience, we set the components $\Phi_{l,m}(u)$ and $\Psi_{l,m}^*(u)$ ($m = 0, \cdots, l$) of the VO's as

$$\widehat{\Phi}_l(u - \frac{1}{2}) = \sum_{m=0}^{l} \Phi_{l,m}(u) \otimes v_m^l,$$

$$\widehat{\Psi}_l^*(u - \frac{k+1}{2})(v_m^l \otimes \cdot) = \Psi_{l,m}^*(u).$$

The fundamental property of the vertex operators is that they satisfy the intertwining relations. In fact, intertwining operators of the algebra could be used to define the Vertex operators in some sense. Here we just pay our attention to the intertwining relations for the highest components $\Phi_{l,l}(v)$ and $\Psi_{l,l}^*(v)$:

$$H^\pm(u)\Phi_{l,l}(v) = \frac{\theta(u - v + \frac{l}{2} \mp \frac{k}{4})}{\theta(u - v - \frac{l}{2} \mp \frac{k}{4})}\Phi_{l,l}(v)H^\pm(u), \tag{5.3}$$

$$E(u)\Phi_{l,l}(v) = \Phi_{l,l}(v)E(u), \tag{5.4}$$

$$F(u)\Phi_{l,l}(v) = \frac{\theta(u - v + \frac{l}{2})}{\theta(u - v - \frac{l}{2})}\Phi_{l,l}(v)F(u); \tag{5.5}$$

$$H^\pm(u)\Psi_{l,l}^*(v) = \frac{\theta^*(u - v - \frac{l}{2} \pm \frac{k}{4})}{\theta^*(u - v + \frac{l}{2} \pm \frac{k}{4})}\Psi_{l,l}^*(v)H^\pm(u), \tag{5.6}$$

$$F(u)\Psi_{l,l}^*(v) = \Psi_{l,l}^*(v)F(u), \tag{5.7}$$

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$$E(u)\Psi^*_{l,l}(v) = \frac{\theta^*(u - v - \frac{l}{2})}{\theta^*(u - v + \frac{l}{2})}\Psi^*_{l,l}(v)E(u). \tag{5.8}$$

It should be noted that all the expressions of the fields in this section are considered to be normal-ordered.

Let us write currents $V^{\pm}(w; r, r^*)$ as:

$$V^+(w; r, r^*) = \exp\{-A_+(l, r^*; 2, k, r|w; \frac{k+2}{2}) - B_+(l, r^*; 1, k, r|w; k+1)\},$$

$$V^-(w; r, r^*) = \exp\{A_-(-l, r; 2, k, r^*|w; \frac{k-2}{2}) + B_-(-l, r; 1, k, r^*|w; -1)\}.$$

Using them and the parameterization given in the second section, we give a new realization of the type *I* and the type *II* VO's in the following theorem:

Theorem 3 . *If we express* $\Phi_{l,l}(v)$ *and* $\Psi^*_{l,l}(v)$ *through:*

$$\Phi_{l,l}(v) = V^+(w; r, r^*)\exp\{A_-(l; 2, k+2|w; \frac{k+2}{2}) + A_+(l; k, k+2|w; \frac{k+2}{2}) + B_+(l; 1, k|w; 1)\}$$

$$\times\exp\{\frac{l\,q_a}{2(k+2)} - \frac{l}{2r}(2p_b + \hat{p})\ln w\},$$

$$\Psi^*_{l,l}(v) = V^-(w; r, r^*)\exp\{-A_-(-l, k+1; 1, k, k+2|w; -\frac{k+2}{2}) + A_+(-l; 2, k+2|w; \frac{k+2}{2})$$

$$-B_-(-l, k+1; 1, 1, k|w; -1) + B_+(-l; 1, 1|w)$$

$$-C_-(-l; 1, 1|w) + C_+(-l; 1, 1|w)\}$$

$$\times\exp\{\frac{l\,q_a}{2(k+2)} + l(q_b + q_c) - l\hat{q} + \frac{l}{2r^*}\hat{p}\ln w\},$$

then they satisfy the intertwining relations (5.3)-(5.8).

Proof: We take the relation (5.3) as an example. In fact, from the bosonic expression of $\Phi_{l,l}(v)$ and (3.18), the following OPE's can be derived:

$$H^+(u)\Phi_{l,l}(v) = q^{l-\frac{kl}{2r}}z^{\frac{l}{r}}\frac{(q^{-l+\frac{k}{2}}\frac{w}{z}; p)_\infty}{(q^{l+\frac{k}{2}}\frac{w}{z}; p)_\infty} : H^+(u)\Phi_{l,l}(v) :,$$

$$\Phi_{l,l}(v)H^+(u) = w^{\frac{l}{r}}\frac{(pq^{-l-\frac{k}{2}}\frac{z}{w}; p)_\infty}{(pq^{l-\frac{k}{2}}\frac{z}{w}; p)_\infty} : H^+(u)\Phi_{l,l}(v) : .$$

The others can be derived similarly. $\square$

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Actually the lower components $\Phi_{l,m}(v)$ and $\Psi^*_{l,m}(v)$ $(m = 0, \cdots, l)$ can be completely determined by the highest ones $\Phi_{l,l}(v)$ and $\Psi^*_{l,l}(v)$, since they obey the following recursive relations:

$$\Phi_{l,m-1}(v) = F^+(v - \frac{l}{2})\frac{\theta(\hat{p} + h + l - m)}{\theta(\hat{p} + h)}\Phi_{l,m}(v) \quad (m = 0, 1, \cdots, l), \tag{5.9}$$

$$\Psi^*_{l,m-1}(v) = \Psi^*_{l,m}(v)E^+(v - \frac{l+k}{2} - r^*)\frac{\theta^*(m)\theta^*(\hat{p} - l + m - 2)}{\theta^*(l - m + 1)\theta^*(\hat{p} - 2)} \quad (m = 0, 1, \cdots, l), \tag{5.10}$$

where $E^+(v)$, $F^+(v)$ are half currents defined by

$$E^+(v) = \varrho^* \oint_{C^*} E(v')\frac{\theta^*(v - v' + k/2 - \hat{p} + 1)\theta^*(1)}{\theta^*(v - v' + k/2)\theta^*(\hat{p} - 1)}\frac{dw'}{2\pi i w'},$$

$$F^+(v) = \varrho \oint_C F(v')\frac{\theta(v - v' + \hat{p} + h - 1)\theta(1)}{\theta(v - v')\theta(\hat{p} + h - 1)}\frac{dw'}{2\pi i w'},$$

and h is one of the Drinfeld generators of $U_q(\widehat{sl_2})$.
Here the contours are

$$C^* : |p^*q^k w| < |w'| < |q^k w|,$$
$$C : |pw| < |w'| < |w|,$$

and the constants ϱ, ϱ^* are chosen to satisfy

$$\varrho\varrho^*\theta^*(1)\frac{\xi(q^{-2}; p^*, q)}{\xi(q^{-2}; p, q)} = q - q^{-1},$$

where the function $\xi(z; p, q)$ is

$$\xi(z; p, q) = \frac{(q^2 z; p, q^4)_\infty (pq^2 z; p, q^4)_\infty}{(q^4 z; p, q^4)_\infty (pz; p, q^4)_\infty}.$$

5.2 The twisted Vertex Operators

In this subsection, we discuss another two vertex operators $\widehat{\Phi}^t_l(u)$ and $\widehat{\Psi}^{*t}_l(u)$ for $U_{q,p}(\widehat{sl_2})$, which are called the twisted type I VO's and the twisted type II VO's (or twisted intertwiners) respectively. Their definitions are analogously to the non-twisted ones. It means that $\widehat{\Phi}^t_l(u)$ and $\widehat{\Psi}^{*t}_l(u)$ are also the operators of the same type as (5.1)-(5.2) and they also have the similar decompositions, with their components denoted as $\Phi^t_{l,m}(u)$ and $\Psi^{*t}_{l,m}(u)$ $(m = 0, \cdots, l)$. However, the crucial

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difference between them lies in that $\widehat{\Phi}_l^t(u)$ and $\widehat{\Psi}_l^{*t}(u)$ satisfy the twisted intertwining relations. Here we also only consider them for the highest components $\Phi_{l,l}^t(v)$ and $\Psi_{l,l}^{*t}(v)$:

$$H^{\pm}(u)\Phi_{l,l}^t(v) = \frac{\theta(u - v + \frac{l}{2} \mp \frac{k}{4})}{\theta(u - v - \frac{l}{2} \mp \frac{k}{4})}\Phi_{l,l}^t(v)H^{\pm}(u), \qquad (5.11)$$

$$E(u)\Phi_{l,l}^t(v) + \Phi_{l,l}^t(v)E(u) = 0, \qquad (5.12)$$

$$F(u)\Phi_{l,l}^t(v) = -\frac{\theta(u - v + \frac{l}{2})}{\theta(u - v - \frac{l}{2})}\Phi_{l,l}^t(v)F(u); \qquad (5.13)$$

$$H^{\pm}(u)\Psi_{l,l}^{*t}(v) = \frac{\theta^*(u - v - \frac{l}{2} \pm \frac{k}{4})}{\theta^*(u - v + \frac{l}{2} \pm \frac{k}{4})}\Psi_{l,l}^{*t}(v)H^{\pm}(u), \qquad (5.14)$$

$$F(u)\Psi_{l,l}^{*t}(v) + \Psi_{l,l}^{*t}(v)F(u) = 0, \qquad (5.15)$$

$$E(u)\Psi_{l,l}^{*t}(v) = -\frac{\theta^*(u - v - \frac{l}{2})}{\theta^*(u - v + \frac{l}{2})}\Psi_{l,l}^{*t}(v)E(u). \qquad (5.16)$$

It is easy to see that for the Cartan parts $H^{\pm}(u)$, there is no difference for the twisted intertwining relations (5.3), (5.11) and the non-twisted ones (5.6), (5.14); while for $E(u)$ and $F(u)$, the difference between them is just a minus sign. With the notations introduced before, we have

$$\Phi_{l,l}^t(v) = V^+(w; r, r^*)\exp\{A_-(k - l + 1; 1, k + 2|w; -\frac{k+2}{2}) + B_-(k - l + 1; 1, 1|w; -1)$$

$$-A_+(k - l, k + 1; 1, k, k + 2|w; \frac{k+2}{2}) - B_+(k - l, k + 1; 1, 1, k|w; 1)$$

$$+C_-(k - l; 1, 1|w) - C_+(k - l; 1, 1|w)\}$$

$$\times \exp\{\frac{(k - l)q_a}{2(k + 2)} + (k - l)(q_b + q_c) + r^*\hat{q} + (\frac{l - r}{r}p_c - \frac{l}{2r}\hat{p})\ln w\},$$

$$\Psi_{l,l}^{*t}(v) = V^-(w; r, r^*)\exp\{A_-(-(k - l); k, k + 2|w; -\frac{k+2}{2}) + B_-(-(k - l); 1, k|w; -1)$$

$$+A_+(l + 2; 2, k + 2|w; \frac{k+2}{2}) + B_+(1|w; l + 1)\}$$

$$\times \exp\{\frac{(k - l)q_a}{2(k + 2)} - (l + r^*)\hat{q} + (-p_b + \frac{l}{2r^*}\hat{p})\ln w\}.$$

Similarly to the non-twisted case, we can prove that $\Phi_{l,l}^t(v)$ and $\Psi_{l,l}^{*t}(v)$ obey the twisted intertwining relations (5.11)-(5.16).

Theorem 4 . $\Phi_{l,l}^t(v)$ and $\Psi_{l,l}^{*t}(v)$ gotten above give a new realization of the twisted type I and the twisted type II VO's. While the lower components $\Phi_{l,m}^t(v)$ and $\Psi_{l,m}^{*t}(v)$ are determined by the same relations (5.9)-(5.10) in which $\Phi_{l,m}(v)$ and $\Psi_{l,m}^*(v)$ are replaced by $\Phi_{l,m}^t(v)$ and $\Psi_{l,m}^{*t}(v)$ respectively.

In fact, there are a few degrees of freedom on choosing the zero-modes of the above non-twisted and twisted vertex operators. Here we only use the simplest ones.

6   Discussion

In this paper, we construct a new free boson realization of $U_{q,p}(\widehat{sl_2})_k$ by twisting the flag manifold realization, which can be viewed as the elliptic deformation of Wakimoto realization. With this approach, the two important objects (screening currents and Intertwiner Operators or vertex operators) are also discussed in details. They all play important roles in calculating correlation functions. Of course the derivation of the multi-point correlation functions is a quiet interesting problem, but in view of its complexity and the length of the manuscript, we will discuss it in the future. Furthermore, it is an interesting problem to extend our results to the other types of Lie algebras.

7   Acknowledgments

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